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NEW FIDELITY CRITERIA FOR DISCRETE-TIME SOURCE ENCODING

JT Rickard (ORINCON Corp. N66001-78-C-0051)
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A probabilistic approach to the source encoding problem requires the specification of a fidelity criterion which measures the degradation of the source output produced by the encoding transformation. Historically, the fidelity criteria generated by difference-type distortion measures have received by far the widest acceptance, due in part to their analytical tractability. This paper introduces two new fidelity criteria for discrete-time sources. These are designated as the "mean incoherence" (MIC) and the "mean log-coherence" (MLC). Both criteria are functionals of the magnitude-squared coherence between the → next page

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source output and its reproduction, and thus do not admit a recognizable time-domain distortion measure representation. Interesting features of these fidelity criteria are indicated, and comparisons are made with the classical mean-squared error (MSE) fidelity criterion. The rate-distortion functions for a stationary Gaussian source subject to constraints on either the MIC or MLC are derived, and the corresponding optimum encoder behavior is described. Finally, the applications where these new fidelity criteria may be advantageous over the MSE criterion are discussed.

SUMMARY

A probabilistic approach to the source encoding problem requires the specification of a fidelity criterion which measures the degradation of the source output produced by the encoding transformation. Historically, the fidelity criteria generated by difference-type distortion measures have received by far the widest acceptance, due in part to their analytical tractability. This paper introduces two new fidelity criteria for discrete-time sources. These are designated as the "mean incoherence" (MIC) and the "mean log-coherence" (MLC). Both criteria are functionals of the magnitude-squared coherence between the source output and its reproduction, and thus do not admit a recognizable time-domain distortion measure representation. Interesting features of these fidelity criteria are indicated, and comparisons are made with the classical mean-squared error (MSE) fidelity criterion. The rate-distortion functions for a stationary Gaussian source subject to constraints on either the MIC or MLC are derived, and the corresponding optimum encoder behavior is described. Finally, the applications where these new fidelity criteria may be advantageous over the MSE criterion are discussed.

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CONTENTS

I. INTRODUCTION

A fundamental prerequisite for source encoding is the selection of an appropriate fidelity criterion for the problem at hand. The criterion chosen should ideally indicate the true average degradation (produced by the encoding transformation) in the utility of the source output from the user's standpoint. This must be the primary concern if the goal is a useful system design. A secondary concern is the analytical tractability of a given criterion. In a practical sense, this may involve nothing more than the ability to estimate the average distortion resulting from a particular encoding operation with a feasible amount of computation. In the rate-distortion theory sense, however, analytical tractability of a fidelity criterion generally implies the ability to obtain either explicit or parametric representations of the rate-distortion function $R(D)$ for a class of sources subject to that fidelity criterion.

The two concerns described above remain major obstacles to the design of practical data compression systems. The general state of affairs as of 1971 was summarized by Berger [1, p. 6] in the introductory chapter of his book:

> This is an extremely important problem that has yet
> to receive anywhere near the attention it merits. In
> the absence of a theoretical machinery for synthesiz-
> ing suitable distortion measures, the emphasis has
> been placed on developing general results for various
> classes of distortion measures. The rationale behind
> this indirect approach is that, as further research
> continues to expand the set of distortion measures
> that have been analyzed successfully, it becomes
> increasingly likely that the system designer can
> find results that are applicable to the particular
> source-user combinations he may encounter.

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Little progress of an analytically satisfying nature has been made in the interim. The system designer who wishes to make use of the results of rate-distortion theory (e.g., $R(D)$ functions and optimum encoder designs) is for the most part constrained to difference-type distortion measures and their corresponding fidelity criteria. The most notable example in this category is of course the ubiquitous mean-squared error (MSE) fidelity criterion. The analytical tractability of this particular criterion has undoubtedly enhanced its wide acceptance. In some applications, however, the MSE and related criteria have little or no physical justification, and can in fact dictate disastrous encoding strategies, as will be demonstrated in a later section. Furthermore, the estimation of MSE and related average distortion values must be done with great care for continuously distributed sources. For example, perfectly acceptable reconstructed waveforms which differ only by a constant scaling factor may yield grossly different MSE distortion values when compared with the original waveforms.

The purpose of this paper is to introduce two new fidelity criteria for discrete-time sources. These will be designated by the terms "mean incoherence" (MIC) and "mean log-coherence" (MLC). Both criteria are functionals of the magnitude-squared coherence (MSC) $\gamma_{xy}^2(\omega)$ between a source $\{x\}$ and its reproduction $\{y\}$. Interesting features of these fidelity criteria are indicated, and comparisons are made with the MSE fidelity criterion. The rate-distortion functions for a stationary Gaussian source subject to constraints on either the MIC or MLC are derived, and the corresponding optimum encoder behavior is described. Finally, the applications where these new fidelity criteria may be advantageous over the MSE criterion are discussed.

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Before proceeding, it is appropriate to clearly define what we mean by the term "fidelity criterion," as opposed to "distortion measure." A distortion measure, in an information-theoretic sense, is a means of assigning a real number to every pair of (finitely or infinitely long) sequences corresponding to a particular source-reconstruction realization. A fidelity criterion, however, assigns a real number to the <u>ensemble</u> of source-reproduction pairs [11, sec. 26], and thus represents an <u>average</u> distortion over the ensemble. The MIC and MLC fidelity criteria dealt with in this paper assume jointly stationary source-reproduction ensembles. As Berger [1, p. 126] points out, this is a reasonable assumption for an ideal system which encodes infinitely long messages from a stationary source, since the system has nothing to gain by operating in a time varying manner. Indeed, the optimum encoding which achieves the rate-distortion limit in many situations requires infinitely long code words. The practical use of the MIC and MLC fidelity criteria implies the estimation of their values from a finite-length span of data (as is the case with MSE measurements). In later sections, we shall briefly mention the manner in which this is done.

II. PRELIMINARIES

We shall begin with the definition and a brief review of certain properties of the MSC function $\gamma_{xy}^2(\omega)$. Proofs of these properties may be found in [2-4].

Definition. The __magnitude-squared coherence__ $\gamma_{xy}^2(\omega)$ between two real, zero-mean, discrete-time, jointly stationary random processes $\{x\}$ and $\{y\}$, possessing auto-spectral densities $\Phi_x(\omega)$ and $\Phi_y(\omega)$, respectively, and cross-spectral density $\Phi_{xy}(\omega)$, is defined by

$$\gamma_{xy}^2(\omega) \triangleq \begin{cases} \dfrac{|\Phi_{xy}(\omega)|^2}{\Phi_x(\omega)\,\Phi_y(\omega)} & \text{for } \Phi_{xy}(\omega) \neq 0, \quad (1) \\[4mm] & \qquad\qquad 0 \leq \omega \leq \pi. \\[2mm] 0 & \text{for } \Phi_{xy}(\omega) = 0, \quad (2) \end{cases}$$

The latter portion (Equation (2)) of this definition allows $\gamma_{xy}^2(\omega)$ to be determined in cases where either $\Phi_x(\omega)$ or $\Phi_y(\omega)$ is zero for some ω. Note that $0 \leq \gamma_{xy}^2(\omega) \leq 1$ for all ω.

Property 2.1. If $\{x\}$ and $\{y\}$ are uncorrelated, then $\gamma_{xy}^2(\omega) \equiv 0$ for all ω.

Property 2.2. If L represents a reversible time-invariant linear operator, and $\{y\} = \{Lx\}$, then $\gamma_{xy}^2(\omega) \equiv 1$ for all ω.

From Properties 2.1 and 2.2, we see that $\gamma_{xy}^2(\omega)$ is a normalized, frequency-domain measure of linear correlation between $\{x\}$ and $\{y\}$, which is generally more informative than the cross-correlation coefficient $\rho_{xy}(\tau)$. The latter quantity is defined by

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$$\rho_{xy}(\tau) \overset{\Delta}{=} \frac{\varphi_{xy}(\tau)}{\varphi_x(0)\,\varphi_y(0)}, \tag{3}$$

where $\varphi_x(\tau)$, $\varphi_y(\tau)$, and $\varphi_{xy}(\tau)$ are the respective auto- and cross-correlation functions of $\{x\}$ and $\{y\}$. Note that $\rho_{xy}(\tau)$ is normalized with respect to the total average power in the two processes, whereas $\gamma_{xy}^2(\omega)$ is normalized for each ω with respect to the auto-spectra of $\{x\}$ and $\{y\}$ at that value of ω. Thus, one may determine the degree of linear correlation in any portion of the spectrum of two processes by examining their MSC function.

<u>Property 2.3</u>. If $\gamma_{xy}^2(\omega) = 1$ for all ω, then there exists a unique time-invariant linear operator L such that $\{y\} = \{Lx\}$ with probability one.

This property suggests the use of $\gamma_{xy}^2(\omega)$ as a frequency-dependent measure of the linearity of a system whose input and output are $\{x\}$ and $\{y\}$, respectively. However, caution must be used when this is contemplated, since $\gamma_{xy}^2(\omega)$ is a function of the particular input process. Clearly, there exist systems which behave linearly with respect to one class of inputs, but not to another (e.g., saturating linear amplifiers). Thus, Property 2.3 is not a sufficient condition for inferring the linearity of a system having $\{x\}$ and $\{y\}$ as its respective input and output.

Note that Property 2.3 can be extended, in the event $\gamma_{xy}^2(\omega) = 1$ for all $\omega \epsilon W \subset [0, \pi]$, to the corresponding "band-restricted" versions of $\{x\}$ and $\{y\}$.

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Property 2.4. If $\{y\} = \{L(x+n)\}$ where $\{x\}$ and $\{n\}$ are uncorrelated, and L is a time-invariant linear operator, then we have

$$\sigma(\omega) \triangleq \frac{\gamma_{xy}^2(\omega)}{1 - \gamma_{xy}^2(\omega)} = \frac{\Phi_x(\omega)}{\Phi_n(\omega)} \ . \tag{4}$$

Property 2.5.   If $\{y\} = \{Tx\}$, where T is any time-invariant oper-
ator (not necessarily linear), then $\sigma(\omega)$ (defined in (4)) is the ratio
of the average power in $\{y\}$ at frequency ω which can be represented
as an optimum (in the mean-square sense) linear estimate of $\{y\}$
given $\{x\}$, to the average power in the residual of this estimate at
frequency ω.

These last two properties suggest the use of $\sigma(\omega)$ as a
frequency-dependent signal-to-additive-noise ratio (SNR), or a
signal-to-quantizing noise ratio (SQR) in cases where $\{y\}$ represents
a quantized version of $\{x\}$.

It should be obvious from the above properties that the MSC
function is useful for statistical comparisons between two processes.
Indeed, it has found wide application in many areas of statistical data
analysis [2-4]. The MSC function is easily estimated using conven-
tional spectrum estimation techniques. We shall now briefly discuss
this problem.

Given two real, zero-mean, discrete-time random processes
$\{u\}$ and $\{v\}$, let the sequences $U_{j\ell}$ and $V_{j\ell}$ represent successive
K-point discrete Fourier transforms (DFT) on segments of a pair
of sample functions from the respective processes, where j is the
time index and ℓ is the frequency index of the transform coefficients.
In practice, one would employ a data window to reduce spectral

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leakage, and possibly overlap successive segments if a limited amount of data were available. Given a finite number of corresponding pairs of DFT coefficients from the two sample functions, and assuming that these are drawn from a jointly ergodic ensemble, we may form an estimate $\hat{\gamma}_{uv}^2(\omega)$ of the MSC at the frequency corresponding to the ℓth coefficient index:

$$\hat{\gamma}_{uv}^2(\ell) = \frac{\left| \sum_{j=1}^{N} U_{j\ell}^* V_{j\ell} \right|^2}{\sum_{j=1}^{N} \left| U_{j\ell} \right|^2 \sum_{j=1}^{N} \left| V_{j\ell} \right|^2},$$

where the "*" above denotes the complex conjugate. Under the assumptions that $\{u\}$ and $\{v\}$ are jointly Gaussian, and that the transform coefficients $U_{j\ell}$ are independent and perfectly windowed (i.e., no spectral leakage), and similarly for the coefficients $V_{j\ell}$, Goodman [5] derived the distribution theory of this estimate. In particular, the estimate is asymptotically unbiased and consistent, with probability density function $p_{\hat{\gamma}}(z)$, given N and a true MSC of γ^2,

$$p_{\hat{\gamma}}(z) = (N-1)(1-\gamma^2)^N (1-z)^{N-2}(1-\gamma^2 z)^{1-2N} {}_2F_1(1-N, 1-N; 1; \gamma^2 z),$$

where $0 \le z \le 1$, and ${}_2F_1(a, b; c; z)$ denotes the hypergeometric function. Several practical aspects of this MSC estimate (e.g., bias, variance, data windowing and overlapping) are discussed in [6, 7]. Hence, the statistical behavior of this estimate is rather well understood.

III. NEW FIDELITY CRITERIA

We now introduce a new application of the MSC function in the area of data compression theory and practice. To this end, we define two fidelity criteria for source encoding, which are functionals of the MSC $\gamma_{xy}^2(\omega)$ between the source $\{x\}$ and its reconstruction $\{y\}$.

Definition. The frequency-weighted <u>mean incoherence</u> (MIC) is given by

$$
\text{MIC} \triangleq \frac{\displaystyle\int_0^\pi d\omega\, W(\omega)\left[1 - \gamma_{xy}^2(\omega)\right]}{\displaystyle\int_0^\pi d\omega\, W(\omega)}, \tag{5}
$$

where $W(\omega)$ $(0 \leq W(\omega) \leq 1)$ is an arbitrary user-specified weighting function. It is tacitly assumed that $W(\omega) = 0$ for all ω such that $\Phi_x(\omega) = 0$, but $W(\omega) \neq 0$ for all ω.

Definition. The frequency-weighted <u>mean log-coherence</u> (MLC) is given by

$$
\text{MLC} \triangleq -\frac{\displaystyle\int_0^\pi d\omega\, W(\omega)\log\gamma_{xy}^2(\omega)}{\displaystyle\int_0^\pi d\omega\, W(\omega)}, \tag{6}
$$

with $W(\omega)$ the same as above.

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The weighting function $W(\omega)$ provides the user with the flexibility of assigning <u>a priori</u> significance to the distortion in different portions of the spectrum.

Fidelity criteria of the type (5) and (6) implicitly assume that the source message and its reconstruction are jointly stationary. Hence, all of the analytic results herein pertain to the encoding of infinitely long messages, and represent lower bounds to that which can be achieved in practice. In order to evaluate the results of encoding a finite length message, one must estimate the MSC between the original and reconstructed waveforms, as described in the previous section. Although these waveforms are inherently nonstationary, due both to their finite duration and the encoder structure, one may still obtain useful results which asymptotically (with increasing code word length) satisfy the assumptions made for the ideal system considered herein. By choosing a block encoding scheme whose block length is an integer multiple of the DFT segment length, the nonstationary transitions between successive code words do not occur within a transform segment, thereby minimizing the effect of finite code word length on the MSC estimate. This estimate is substituted into Equations (5) and (6), and the integrals evaluated numerically, to obtain estimated values of the MIC and MLC. These values are subject to two (coupled) sources of error: (1) the statistical error in the MSC estimates; and (2) the error resulting from numerical integration. Since our present purpose is to derive the rate-distortion properties of the MIC and MLC fidelity criteria, we will not enter into a detailed error analysis of MIC and MLC estimates in this paper. Briefly, however, we note that the first source of error can be made arbitrarily small by increasing N, the number of DFT segments averaged in the MSC

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estimate, since (see [7]) both the bias and the variance of this estimate are $O\left(\frac{1}{N}\right)$. The second source of error can be made arbitrarily small by increasing K, and choosing N such that as K → ∞, $\frac{K}{N}$ → 0. Hence, with an adequate amount of source-reconstruction data, one can evaluate the MIC and MLC to any degree of accuracy.

The following two properties are consequences of Properties 2.1–2.3 of the previous section.

<u>Property 3.1.</u> The MIC distortion ranges from zero to unity, with zero distortion corresponding to the case {y} = {Lx} (L a reversible, time-invariant linear operator) with probability one, and unity distortion resulting when {x} and {y} are uncorrelated.

<u>Property 3.2.</u> The MLC distortion ranges from zero to infinity, with zero distortion corresponding to the case {y} = {Lx} with probability *one*, and infinite distortion resulting when $\gamma_{xy}^{2}(\omega)$ is zero over any non-zero-measure subset of $[0, \pi]$.

Thus we observe a fundamental difference between the MIC and MLC criteria and the MSE criterion, viz., any reversible linear transformation of the source sequence {x} affected by a data compression system results in zero MIC or MLC distortion, whereas such a transformation may result in arbitrary MSE distortion. The scaling problem inherent in evaluating the MSE between two sequences is not present with either the MIC or MLC criterion. Furthermore, since a number of signal processing operations (e.g., beamforming, spectral detection) are invariant to certain classes of linear prefiltering, there is no justification for insisting on an MSE type of fidelity constraint by the user. Of course, no

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data compression system will affect a perfect linear transformation of the source, and any nonlinear operation (e.g., quantizing) will be reflected in a reduced coherence between $\{x\}$ and $\{y\}$, with a resulting non-zero distortion.

Property 3.3. The distortion spectrum of the MIC (i.e., the numerator integrand of (5)) represents, for each value of ω, the (weighted) fraction of $\Phi_y(\omega)$ which is due to $\Phi_e(\omega)$, the auto-spectral density of the process $\{y - \hat{y}\}$, where $\{\hat{y}\}$ is the optimum linear estimate of $\{y\}$ given $\{x\}$.

This follows from Properties 2.4–2.5 of the previous section, and lends physical significance to the MIC fidelity criterion. Thus the net value of the MIC, given by (5), represents the (weighted) average fraction of the total power in $\{y\}$ which is due to the error process $\{y - \hat{y}\}$.

Property 3.4. The distortion spectrum of the MLC (i.e., the numerator integrand of (6)) represents, for each value of ω, the weighted negative logarithm of the fraction of $\Phi_y(\omega)$ which is due to $\Phi_{\hat{y}}(\omega)$, the auto-spectral density of the process $\{\hat{y}\}$ defined above.

Again, we see the similar physical significance of the MLC fidelity criterion. The user would choose the MLC as his fidelity criterion if he were unwilling to tolerate the total elimination from transmission of any (non-zero-measure) portion of the source spectrum $\Phi_x(\omega)$ (excepting portions where $W(\omega) = 0$), since this would result in infinite distortion.

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Before proceeding to the next section, we mention one caution which should be observed when using the MIC and MLC fidelity criteria. As previously noted, neither criterion penalizes a reversible linear transformation of the source output $\{x\}$. In situations where some such transformations may be undesirable with regard to the final output to the user, one must transmit the additional side information needed to invert the first transformation. This would seldom seem to be a problem in practice, since one would not deliberately design a source encoder which implemented an undesirable linear transformation of the source output.

IV. RATE-DISTORTION FUNCTIONS FOR GAUSSIAN SOURCES

In this section, we derive the rate-distortion functions for a discrete-time stationary Gaussian source under either MIC or MLC fidelity constraints. We shall make use of the following results.

Definition. The <u>average mutual information rate</u> $I(x; y)$ between two discrete-time, jointly stationary random processes $\{x\}$ and $\{y\}$ possessing joint and marginal n-th order multivariate density functions $p_{xy}^{(n)}(\underline{u}, \underline{v})$, $p_x^{(n)}(\underline{u})$, and $p_y^{(n)}(\underline{v})$, respectively ($\underline{u}$ and $\underline{v}$ are n-dimensional vector variables of successive source values), is defined by

$$I(x; y) = \lim_{n \to \infty} \frac{1}{n} I_n \left(X^{(n)}; Y^{(n)} \right) , \qquad (7)$$

where

$$I_n \left(X^{(n)}; Y^{(n)} \right) \triangleq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\underline{u} \, d\underline{v} \, p_{xy}^{(n)}(\underline{u}, \underline{v}) \log \left[\frac{p_{xy}^{(n)}(\underline{u}, \underline{v})}{p_x^{(n)}(\underline{u}) \, p_y^{(n)}(\underline{v})} \right] . \qquad (8)$$

The above definition is sufficient for the Gaussian sources to be considered in this section. For a more general definition, see [1, Ch. 7].

Theorem 4.1 (Pinsker). Let $\{x\}$ be a discrete-time Gaussian process which is jointly stationary with another discrete-time process $\{y\}$, the two processes possessing auto-spectral densities $\Phi_x(\omega)$ and $\Phi_y(\omega)$, respectively, and cross-spectral density $\Phi_{xy}(\omega)$, $0 \leq \omega \leq \pi$. Then the greatest lower bound to the average mutual information rate $I(x; y)$ is obtained when $\{x\}$ and $\{y\}$ are jointly Gaussian.

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<u>Proof</u>: See [8] or [9].

<u>Theorem 4.2.</u> Let $\{x\}$ and $\{y\}$ be zero-mean, jointly stationary Gaussian processes. Then there exists a linear operator L and a stationary Gaussian process $\{z\}$ independent of $\{x\}$ such that

$$y_k = L x_k + z_k , \tag{9}$$

where x_k, y_k, and z_k are typical realizations of $\{x\}$, $\{y\}$, and $\{z\}$, respectively.

<u>Proof</u>: See [1, p. 125].

<u>Theorem 4.3 (Pinsker)</u>. Let $\{x\}$ and $\{y\}$ be discrete-time, jointly stationary Gaussian processes possessing respective auto- and cross-spectral densities $\Phi_x(\omega)$, $\Phi_y(\omega)$, and $\Phi_{xy}(\omega)$. Then their average mutual information rate $I(x;y)$ is given by

$$I(x;y) = -\frac{1}{2\pi} \int_0^\pi d\omega \log\left[1 - \gamma_{xy}^2(\omega)\right]. \tag{10}$$

<u>Proof</u>: See [9, p. 175].

With the above preliminaries, we shall now determine the rate-distortion functions $R_I(D)$ and $R_L(D)$ of a Gaussian source $\{x\}$ under MIC and MLC fidelity constraints, respectively. Specifically, the fidelity requirement is of the form

$$\frac{\int_0^\pi d\omega\, K(\omega)}{\int_0^\pi d\omega\, W(\omega)} \le D , \tag{11}$$

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where for $R_I(D)$, $K(\omega)$ is given by

$$K(\omega) = K_I(\omega) \triangleq W(\omega) \left[1 - \frac{|\Phi_{xy}(\omega)|^2}{\Phi_x(\omega)\,\Phi_y(\omega)} \right], \tag{12}$$

and for $R_L(D)$, $K(\omega)$ is given by

$$K(\omega) = K_L(\omega) \triangleq -W(\omega) \log \left[\frac{|\Phi_{xy}(\omega)|^2}{\Phi_x(\omega)\,\Phi_y(\omega)} \right]. \tag{13}$$

By our convention, $\{y\}$ is taken to be the reconstruction of $\{x\}$.

The rate-distortion function $R(D)$ of a source $\{x\}$ with respect to such a fidelity requirement is defined by

$$R(D) = \inf_{Q(D)} I(x; y), \tag{14}$$

where the infimum is taken over the class $Q(D)$ of all conditional probability measures for which (11) is satisfied. However, since (11) is expressed solely in terms of the spectral densities $\Phi_x(\omega)$, $\Phi_y(\omega)$, and $\Phi_{xy}(\omega)$, we conclude from Theorem 4.1 that the minimum rate in (14) occurs when $\{x\}$ and $\{y\}$ are jointly Gaussian. Therefore, by Theorem 4.3, the optimization problem implied by (14) is equivalent to minimizing the quantity

$$I(x; y) = -\frac{1}{2\pi} \int_0^\pi d\omega \, \log \left[1 - \frac{|\Phi_{xy}(\omega)|^2}{\Phi_x(\omega)\,\Phi_y(\omega)} \right], \tag{15}$$

subject to the constraint (11).

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The above minimization problem may be approached via the calculus of variations [10]. Following an argument similar to Berger's [1, pp. 126-129], note that since the optimum $\{y\}$ is such that $\{x\}$ and $\{y\}$ are jointly Gaussian, we may express $\Phi_y(\omega)$, by Theorem 4.2, as

$$\Phi_y(\omega) = |B(\omega)|^2 \Phi_x(\omega) + \Phi_z(\omega) , \tag{16}$$

which implies that

$$\Phi_y(\omega) - |B(\omega)|^2 \Phi_x(\omega) \geq 0 , \tag{17}$$

by virtue of the non-negativity of $\Phi_z(\omega)$. In these equations, $B(\omega)$ is the transfer function of an as yet undetermined linear operator. In addition, we have by Theorem 4.2

$$\Phi_{xy}(\omega) = B(\omega) \Phi_x(\omega) . \tag{18}$$

Thus the kernels $K_I(\omega)$ and $K_L(\omega)$ may be written as

$$K_I(\omega) = W(\omega) \left[1 - \frac{|B(\omega)|^2 \Phi_x(\omega)}{|B(\omega)|^2 \Phi_x(\omega) + \Phi_z(\omega)} \right] , \tag{19}$$

$$K_L(\omega) = - W(\omega) \log \left[\frac{|B(\omega)|^2 \Phi_x(\omega)}{|B(\omega)|^2 \Phi_x(\omega) + \Phi_z(\omega)} \right] . \tag{20}$$

Substituting (16) and (18) into (15), and combining with (11), we see that the minimization problem reduces to finding the critical point of the functional

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$$J = \int_0^\pi d\omega \left\{ \log \left(\frac{\Phi_y(\omega) - |B(\omega)|^2 \Phi_x(\omega)}{\Phi_y(\omega)} \right) - \lambda K(\omega) \right\} , \quad (21)$$

where λ (≥ 0) is a Lagrange multiplier, and $K(\omega)$ is replaced by $K_I(\omega)$ (Equation (19)) for the MIC fidelity criterion, or by $K_L(\ )$ (Equation (20)) for the MLC fidelity criterion. Since $\Phi_x(\omega)$ is fixed, the independent variables in (21) are the functions $\Phi_y(\omega)$, $B_1(\omega) \overset{\Delta}{=} \text{Re} \{B(\omega)\}$, and $B_2(\omega) = \text{Im} \{B(\omega)\}$. Setting the variation of J with respect to each of these three functions equal to zero yields a single equation, an indication that no unique values of $B_1(\omega)$ and $B_2(\omega)$ are required in conjunction with the optimum $\Phi_y(\omega)$.[1] We now consider each fidelity criterion separately.

MIC Fidelity Criterion. The variational equations in this case reduce to

$$\Phi_y(\omega) = \left(1 - \frac{1}{\lambda W(\omega)} \right)^{-1} |B(\omega)|^2 \Phi_x(\omega) . \quad (22)$$

In order to satisfy (17), we require that

$$\left(1 - \frac{1}{\lambda W(\omega)} \right)^{-1} \geq 1 , \quad (23)$$

else $|B(\omega)|^2 = 0$ and hence $\Phi_y(\omega) = 0$ for all ω where this does not hold true. The optimum MSC $\gamma_{xy}^2(\omega)$ for minimum-rate transmission is then given by

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[1] This can be shown to be a consequence of the invariance of the MSC function to time-invariant linear transformations of either $\{x\}$ or $\{y\}$.

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$$\hat{\gamma}_{xy}^2(\omega) = \frac{|B(\omega)|^2 \Phi_x(\omega)}{\Phi_y(\omega)} = \max\left(0, 1 - \frac{1}{\lambda W(\omega)}\right). \qquad (24)$$

Having determined the above function, we need only to substitute it into (15) to obtain $R_I(D)$, and thus we have the following result.

$\underline{R_I(D) \text{ for a Gaussian Source}}$. The rate-distortion function $R_I(D)$ for a stationary Gaussian source subject to a frequency-weighted MIC fidelity requirement is given parametrically by the equations

$$R_I(D) = \frac{1}{2\pi} \int_0^\pi d\omega \, \max\left[0, \log(\lambda W(\omega))\right], \qquad (25)$$

and

$$D = \frac{\displaystyle\int_0^\pi d\omega \, W(\omega) \min\left[1, \frac{1}{\lambda W(\omega)}\right]}{\displaystyle\int_0^\pi d\omega \, W(\omega)}. \qquad (26)$$

In particular, if $W(\omega)$ is unity for all ω (i.e., no de-emphasis is placed on the distortion in any portion of the spectrum), then $\lambda = D^{-1}$, and we obtain the simple result

$$R_I(D) = -\frac{1}{2} \log D, \quad 0 \le D \le 1. \qquad (27)$$

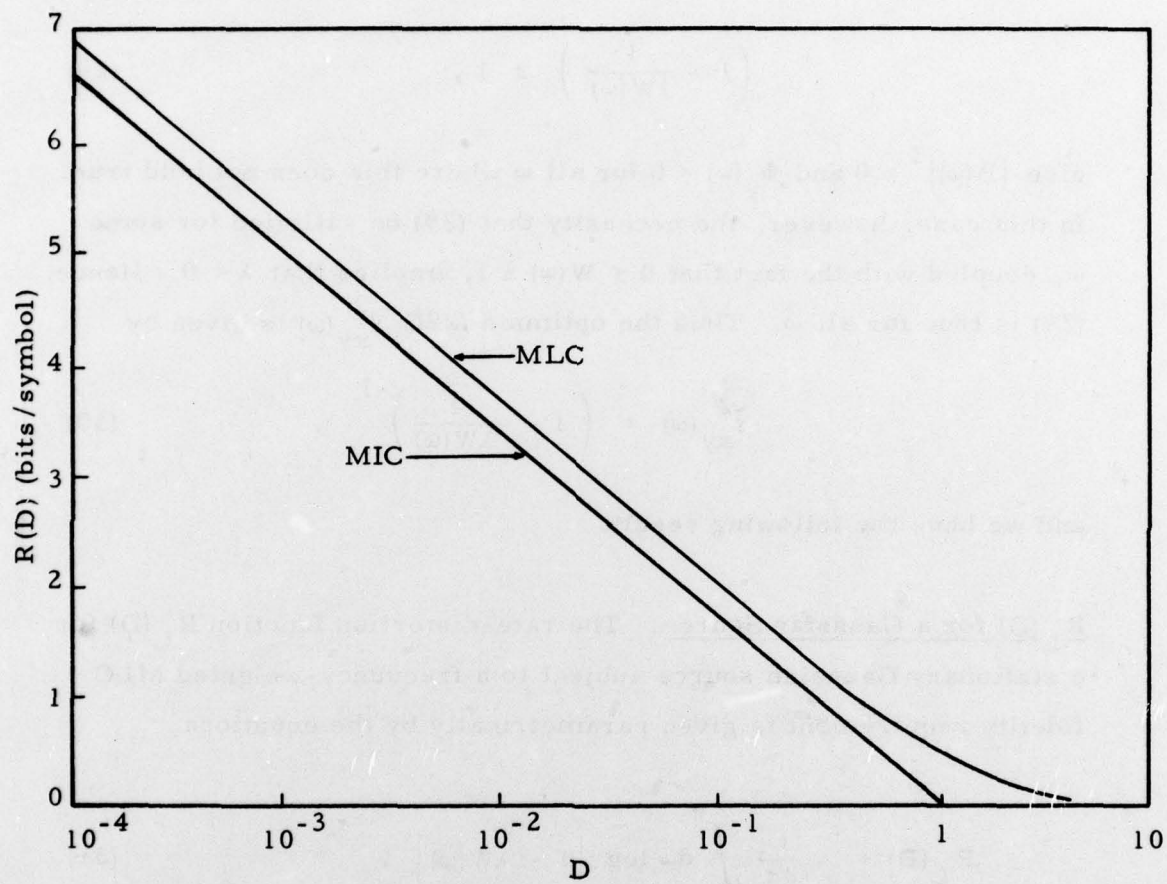This function is plotted in Figure 1 for a base 2 logarithm.

Figure 1. Gaussian MIC and MLC R(D) curves.

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MLC Fidelity Criterion. The variational equations reduce to

$$\Phi_y(\omega) = \left(1 - \frac{1}{\lambda W(\omega)} \right) |B|^2 \Phi_x(\omega) ,\qquad (28)$$

where (17) now requires

$$\left(1 - \frac{1}{\lambda W(\omega)} \right) \geq 1 ,\qquad (29)$$

else $|B(\omega)|^2 = 0$ and $\Phi_y(\omega) = 0$ for all ω where this does not hold true. In this case, however, the necessity that (29) be satisfied for some ω, coupled with the fact that $0 \leq W(\omega) \leq 1$, implies that $\lambda < 0$. Hence, (29) is true for all ω. Thus the optimum MSC $\gamma_{xy}^2(\omega)$ is given by

$$\tilde{\gamma}_{xy}^2(\omega) = \left(1 - \frac{1}{\lambda W(\omega)} \right)^{-1} ,\qquad (30)$$

and we have the following result.

R$_L$(D) for a Gaussian Source. The rate-distortion function $R_L(D)$ for a stationary Gaussian source subject to a frequency-weighted MLC fidelity requirement is given parametrically by the equations

$$R_L(D) = \frac{1}{2\pi} \int_0^\pi d\omega \log \left[1 - \lambda W(\omega) \right] ,\qquad (31)$$

and

$$D = \frac{\displaystyle\int_0^\pi d\omega\, W(\omega) \log \left(1 - \frac{1}{\lambda W(\omega)} \right)}{\displaystyle\int_0^\pi d\omega\, W(\omega)} .\qquad (32)$$

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If $W(\omega)$ equals unity for all ω, then $\lambda = \left(1 - b^D\right)^{-1}$ (b is the base of the logarithm), and

$$R_L(D) = -\frac{1}{2} \log \left(1 - b^{-D}\right), \quad 0 \le D \le \infty . \qquad (33)$$

This function is also plotted in Figure 1 for a base 2 logarithm.

The results of this section suggest the operation of an optimum encoder for Gaussian sources. Let us assume for the moment that the weighting function $W(\omega)$ is unity for all ω, as this case is somewhat easier to visualize. Then from Equations (24) and (30), we see that the minimum transmission rate under either MIC or MLC fidelity constraints is obtained when the MSC $\gamma_{xy}^2(\omega)$ is uniform for all ω, its specific value determined by the allowable distortion D. Referring back to Properties 2.4 and 2.5, a uniform value of $\gamma_{xy}^2(\omega)$ implies a uniform value of SQR $\sigma(\omega)$ for all ω. Thus, from Equation (4), the error spectrum produced by an ideal encoder will be a scaled copy of the original source spectrum, as shown in Figure 2. The achievement of such an ideal system would, of course, require an infinitely long block code.
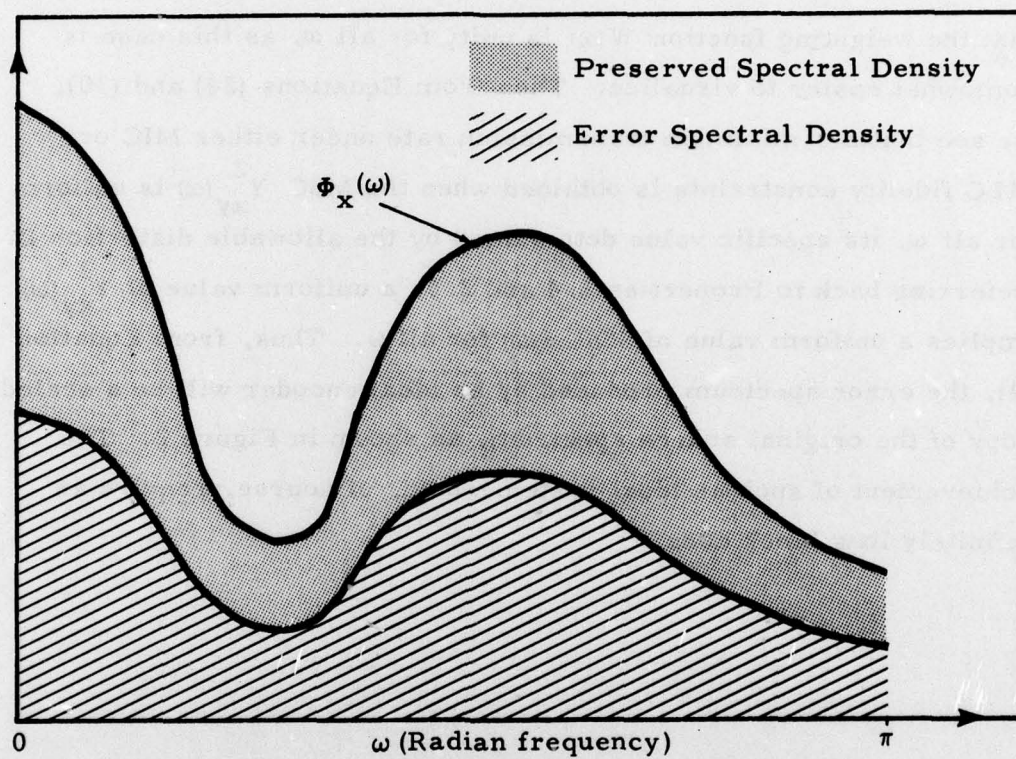
Figure 2. Illustration of preserved source and error spectral densities for optimum MIC or MLC encoding.

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V. COMPARISON WITH MSE SOURCE ENCODING

It is instructive to compare the results of the previous section with the corresponding results for the encoding of Gaussian sources under a MSE fidelity criterion [1]. The rate-distortion function $R_E(D)$ in the latter case is given parametrically by

$$R_E(D) = \frac{1}{2\pi} \int_0^\pi d\omega \, \max \left(0, \log \frac{\Phi_x(\omega)}{\theta} \right), \tag{34}$$

and

$$D = \frac{1}{\pi} \int_0^\pi d\omega \, \min \left(\theta, \Phi_x(\omega) \right). \tag{35}$$

In the event that $\Phi_x(\omega) = 1$ for all ω, i.e., $\{x\}$ is a unity-variance, memoryless source, then we have

$$R_E(D) = -\frac{1}{2} \log D, \quad 0 \leq D \leq 1. \tag{36}$$

Comparing this result with (27), we note that the $R_I(D)$ curve is identical to the $R_E(D)$ curve for this source. However, the $R_I(D)$ curve is applicable to <u>all</u> Gaussian sources regardless of the magnitude or the shape of their auto-spectral density $\Phi_x(\omega)$, whereas the $R_E(D)$ curve for unity-variance sources with non-white spectrum lies below the curve in (36). This behavior points out another fundamental difference between the MSE and the MIC (or MLC) fidelity criteria. As illustrated in Figure 3 (similar to Berger's [1], p. 122), the optimum MSE encoding strategy reproduces only those portions of the source
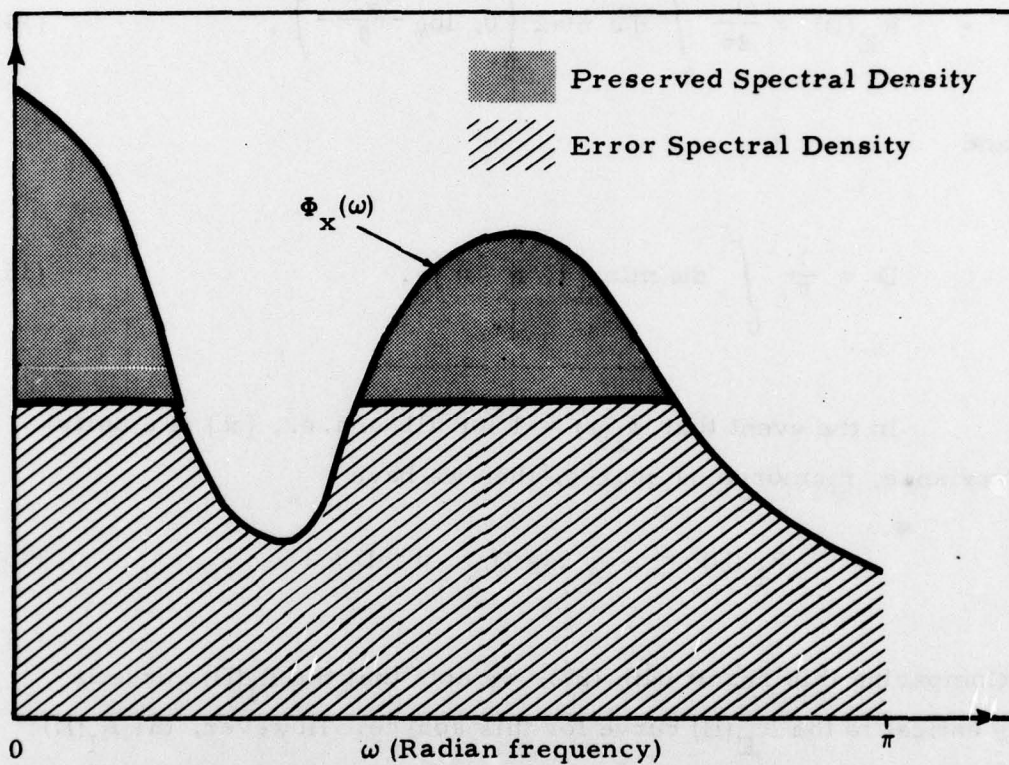
25

Figure 3. Illustration of preserved source and error spectral densities for optimum MSE encoding (see Berger [1], p. 122).

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spectrum $\Phi_x(\omega)$ such that $\Phi_x(\omega) > \theta$, where the fraction $(\Phi_x(\omega) - \theta)/$ $\Phi_x(\omega)$ of the source power in an infinitesimal band centered at ω is preserved in the reproduction $\{y\}$. In general, a different $R_E(D)$ curve results for each different source spectrum. However, referring again to Figure 2, the optimum MIC (or MLC) encoder reproduces all portions of the source spectrum $\Phi_x(\omega)$ with the same relative fidelity. Thus, the savings in required transmission rate (for fixed D) afforded by the MSE fidelity criterion for sources with memory is not available with the MIC or MLC fidelity criteria. A moment's reflection will convince one that the aforementioned savings is purchased at the expense of higher relative distortion in "quieter" portions of the spectrum.

If $\Phi_x(\omega)$ in (34) and (35) is replaced by $|A(\omega)|^2 \Phi_x(\omega)$, we obtain the rate-distortion function for a Gaussian source subject to a frequency-weighted MSE (FWMSE) fidelity criterion [1]. The particular choice

$$|A(\omega)|^2 = \begin{cases} \left[\Phi_x(\omega)\right]^{-1}, & \Phi_x(\omega) > 0, \\ \\ 0, & \Phi_x(\omega) = 0, \end{cases} \tag{37}$$

makes these equations identical to (25) and (26) (for the case $W(\omega) \equiv 1$), i.e., the rate-distortion function $R_I(D)$ for a Gaussian source subject to the MIC fidelity criterion is identical to the rate-distortion function of the same source subject to a FWMSE fidelity criterion, where the weighting function is the inverse of the source spectrum. It should be emphasized, however, that the two fidelity criteria are not equivalent. This FWMSE is still sensitive to a linear transformation between the source and its reconstruction,

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while the MIC is not. Thus, the encoder-decoder structure dictated by this FWMSE criterion is more rigid than that specified by the MIC criterion.

As an example of an application in which the MIC or MLC fidelity criteria would be preferred over the MSE criterion, we consider the problem of transmitting the (Gaussian) output {x} of a remote passive surveillance sensor, where the reconstructed data is to be (auto- or cross-) spectrum analyzed. For example, the user may be interested in detecting the presence of narrow-bandwidth spectral components against a background of broad-bandwidth (generally non-white) noise. The results of optimum encoding of such a source under MSE and MIC (or MLC) fidelity criteria are depicted in Figures 4(a) and 4(b), respectively. The MSE encoding produces relatively little distortion of the narrow-band components which lie in the "louder" portions of the spectrum, while totally eliminating components lying in the quietest portion of the spectrum. It should be obvious that such an encoding strategy may well be disastrous from the users' point of view. From the results of the previous section, however, optimum MIC or MLC encoding produces a reconstruction process {y} which is related to the source process {x} via Equation (9):

$$y_k = L x_k + z_k \; , \tag{38}$$

where {z} is a Gaussian process independent of {x}. Moreover, from Property 2.4, the fraction of $\Phi_y(\omega)$ which is contributed by the source process is constant for all ω (for $W(\omega) \equiv 1$). Therefore, if the reconstruction process is subjected to auto-spectrum analysis, the relative error contributed by the noise {z} is constant at all
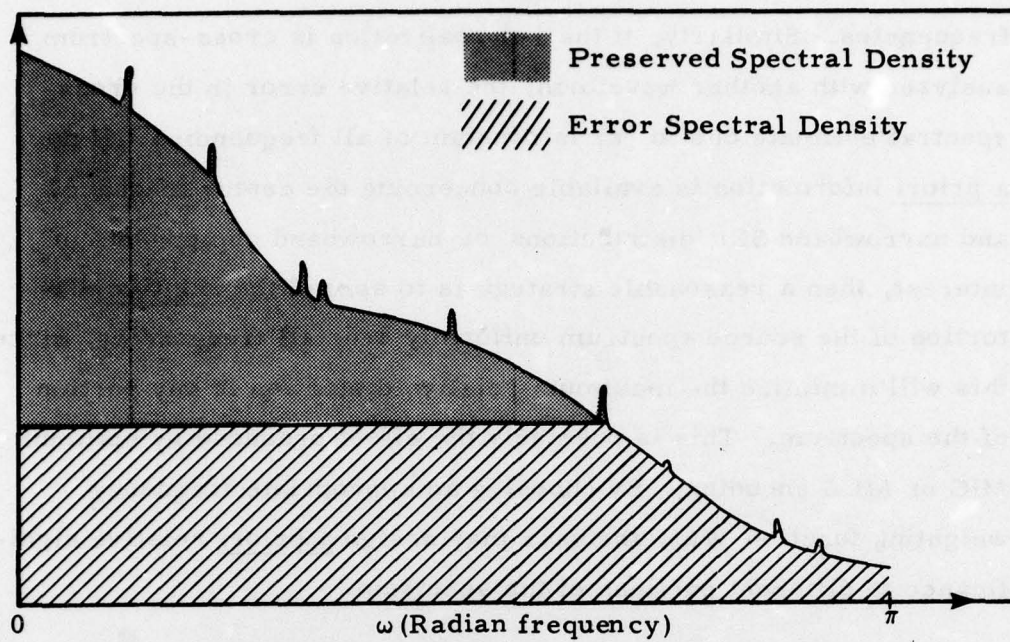
28

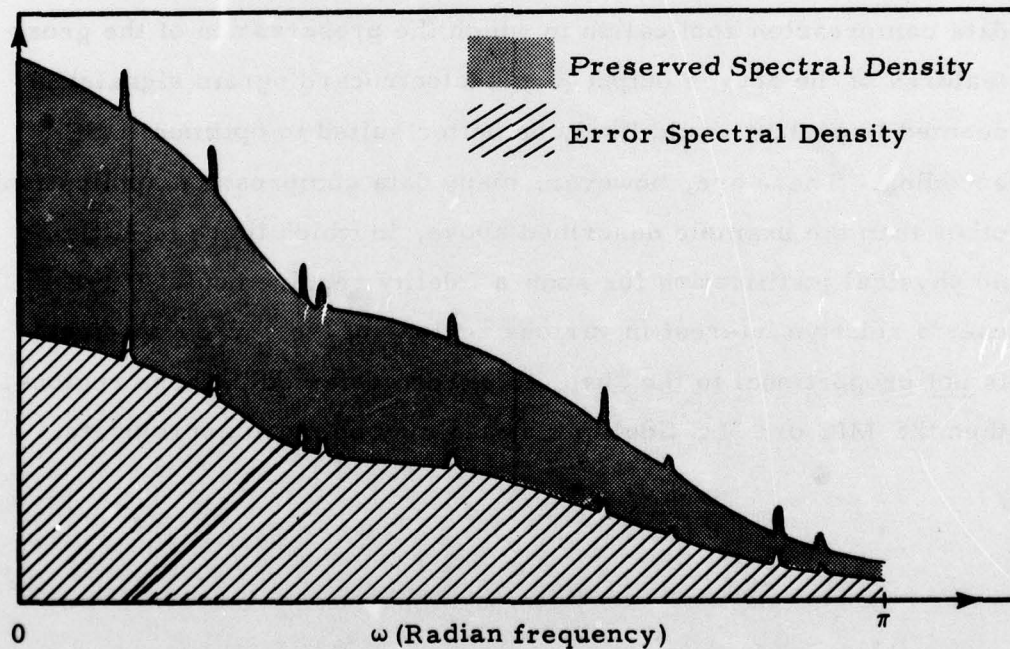Figure 4(a). Example of optimum MSE encoding effects.

Figure 4(b). Example of optimum MIC or MLC
encoding effects.

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frequencies. Similarly, if the {y} realization is cross-spectrum analyzed with another waveform, the relative error in the cross-spectral estimate due to {z} is constant at all frequencies. If no a priori information is available concerning the center frequency and narrowband SNR distributions of narrowband components of interest, then a reasonable strategy is to spread the relative distortion of the source spectrum uniformly over all frequencies, since this will minimize the maximum relative distortion in any portion of the spectrum. This is precisely the effect produced by optimum MIC or MLC encoding. By choosing an appropriate frequency weighting function $W(\omega)$, the user may assign a priori relative significance to different portions of the spectrum.

No claim is made that the MIC or MLC fidelity criteria are superior to the MSE criterion for all applications. Indeed, any data compression application in which the preservation of the gross features of the source output (e.g., electrocardiogram signals) is deemed important would likely be better suited to optimum MSE encoding. There are, however, many data compression applications, other than the example described above, in which there is little or no physical justification for such a fidelity requirement. If the user's relative interest in various regions of the source spectrum is not proportional to the absolute power contained in these regions, then the MIC or MLC fidelity criteria may be more appropriate.

VI. CONCLUSION

Two new fidelity criteria for discrete-time source encoding have been introduced. Interesting features of the MIC and MLC fidelity criteria have been described, and a comparison made with the mean-squared error (MSE) fidelity criterion. The rate-distortion functions for a stationary Gaussian source subject to MIC or MLC fidelity constraints were derived, and the corresponding optimum encoder behavior explained. These results indicate a fundamental difference between the MSC-related fidelity criteria and the MSE criterion. The remote passive surveillance problem represents an application where this difference is highly significant insofar as the optimum encoding strategies are concerned.

Our current efforts are addressing the problem of practical encoder designs for these new fidelity criteria, including the performance of these encoders relative to the rate-distortion curves which have been derived.

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